

ON THE FINITE-DIMENSIONAL DYNAMICAL SYSTEMS WITH LIMITED COMPETITION

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ABSTRACT. The asymptotic behavior of dynamical systems with limited competition is investigated. We study index theory for fixed points, permanence, global stability, convergence everywhere and coexistence. It is shown that the system has a globally asymptotically stable fixed point if every fixed point is hyperbolic and locally asymptotically stable relative to the face it belongs to. A nice result is the necessary and sufficient conditions for the system to have a globally asymptotically stable positive fixed point. It can be used to establish the sufficient conditions for the system to persist uniformly and the convergence result for all orbits. Applications are made to time-periodic ordinary differential equations and reaction-diffusion equations.

1. INTRODUCTION

Recently, an approach has been found to competing dynamical systems with two species. Hess and Lazer [1] did the first work in this area. Hsu, Smith and Waltman [2], Hsu, Waltman and Ellermeyer [3], Smith and Thieme [4] and references therein continued this research.

Let X_1 and X_2 be ordered Banach spaces with positive cones X_1^+ and X_2^+ such that $\text{Int}X_i^+ \neq \emptyset$ for $i = 1, 2$, and let the order in both of these spaces be denoted by “ \leq ”. We define an order \leq_K in $X_1 \times X_2$ as follows:

$$(x_1, x_2) \leq_K (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } y_2 \leq x_2, \text{ where } K = X_1^+ \times (-X_2^+).$$

The coordinates of a point $(x_1, x_2) \in X_1^+ \times X_2^+$ are viewed as representing the population density of two species in competition with each other. A *competitive* system is a map $T = (T_1, T_2) : X_1^+ \times X_2^+ \rightarrow X_1^+ \times X_2^+$ with the properties that an increase in x_1 coupled with a decrease in x_2 results in an increase in T_1 and a decrease in T_2 and that, symmetrically, a decrease in x_1 coupled with an increase in x_2 results in a decrease in T_1 and an increase in T_2 . These natural properties imply that T preserves the order \leq_K , that is, $T(x_1, x_2) \leq_K T(y_1, y_2)$ whenever $(x_1, x_2) \leq_K (y_1, y_2)$.

Hess and Lazer [1] first established this abstract model, developed the theory of exclusion and coexistence, and applied it to reaction-diffusion equations with Neumann boundary conditions. Hsu, Smith and Waltman [2] took a more topological approach, relaxing the smoothness hypotheses used by Hess and Lazer but

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retaining a strong compactness hypothesis and the assumption that the positive cones have nonempty interior. Very recently, Smith and Thieme [4] have been able to reduce the compactness hypothesis and, in most cases, to drop the assumption that the positive cones have nonempty interior. They have derived the conditions for stable coexistence, bi-stability and competitive exclusion, and presented a complete classification of all possible outcomes in the case that there is at most one positive fixed point representing coexistence of both species. All the results in the above-mentioned papers can only be applied to systems with exactly two species.

It is obvious that a competitive system with two species can be generalized to systems with more than two species. Precisely, letting X_i be ordered Banach spaces with positive cones X_i^+ having nonempty interior for $i = 1, 2, \dots, n$ ($n \geq 2$), and letting $1 \leq k < n$, we get a cone $K = X_1^+ \times \dots \times X_k^+ \times (-X_{k+1}^+) \times \dots \times (-X_n^+)$ in the space $X = \prod_{i=1}^n X_i$. The two competing species are replaced by two competing subcommunities, which are labelled by $I = \{1, \dots, k\}$ and $J = \{k+1, \dots, n\}$. $T: \prod_{i=1}^n X_i^+ \rightarrow \prod_{i=1}^n X_i^+$ is called a system with *limited competition* if $T(x) \leq_K T(y)$ whenever $x \leq_K y$, where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \prod_{i=1}^n X_i^+$. If the number of species is two, then this defined system is just the competitive one described in [1]–[4]. However, when the number of species is greater than two, such maps characterize those systems with the properties that “friends of friends are friends,” “friends of enemies are enemies,” and “enemies of enemies are friends.” For systems of ordinary differential equations, such models can be described in the following form:

$$(1.1) \quad \dot{x}_i = x_i f_i(x_1, x_2, \dots, x_n, t), \quad x_i \geq 0, \quad 1 \leq i \leq n.$$

The Jacobians with respect to x of $f = (f_1, \dots, f_n)$ have the form

$$(1.2) \quad D_x f(x, t) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

where A is a $k \times k$ matrix, B is a $k \times (n - k)$ matrix, C is an $(n - k) \times k$ matrix, D is an $(n - k) \times (n - k)$ matrix, and each off-diagonal element of A and D is nonnegative, and B and C are nonnegative matrices.

Although Takeuchi, Adachi and Tokumaru [5] and Travis and Post [6] first considered this class of systems in an ecological context, and many others continued to study them using techniques of mathematical programming, Smith [7, 8] began to investigate the asymptotic behavior of solutions of these general equations systematically in a way consistent with the spirit described above and the theory of monotone dynamical systems pioneered by Hirsch [9]–[11].

The work in [5]–[8] focuses on autonomous systems. Smith [7] determined the asymptotic behavior of (1.1) in the case that (1.2) holds, and obtained sufficient conditions for all species to persist uniformly in the sense of Butler, Waltman and Freedman [12]–[14]. He also sharpened this result for the Lotka-Volterra system

$$(1.3) \quad \dot{x}_i = x_i \left(r_i + \sum_{j=1}^n a_{ij} x_j \right), \quad x_i \geq 0, \quad 1 \leq i \leq n,$$

and proved a result on coexistence in a globally stable positive equilibrium in the case that $M = (a_{ij})_{n \times n}$ has the form (1.2) and is stable. For the Lotka-Volterra

system (1.3), we should mention a remarkable result of Takeuchi and Adachi [15], in which they proved that there is an equilibrium attracting all positive initial points if M is stable. More general systems were considered in [16, 17], where more elegant results were obtained. We observe that (1.1) has the property

(H1) all faces of R_+^n making up the boundary of R_+^n are invariant for (1.1), and that if M is stable then (1.3) satisfies

(H2) every fixed point (equilibrium) is hyperbolic, and

(H3) every fixed point is locally asymptotically stable relative to the face to which it belongs.

The purpose of this paper is to study the asymptotic behavior of finite-dimensional discrete-time dynamical systems $\{T^n\}_{n \geq 0}$ with limited competition. Under the assumptions (H1) through (H3), we shall prove that there exists a fixed point p which attracts all positive initial points. Moreover, we shall show that the two subcommunities coexist in a globally stable positive fixed point if and only if

(1) there is an index subset $L \subset N = \{1, 2, \dots, n\}$ with $L \supset I = \{1, 2, \dots, k\}$ such that the face $H_L^+ = \{x : x_i \geq 0 \text{ for all } i \in L \text{ and } x_k = 0 \text{ for all } k \notin L\}$ has a positive fixed point u satisfying $\frac{\partial T_i}{\partial x_j}|_u > 1$ for any $j \notin L$, and

(2) there is an index subset $P \supset J = \{k+1, k+2, \dots, n\}$ such that the face H_P^+ has a positive fixed point v satisfying $\frac{\partial T_i}{\partial x_i}|_v > 1$ for any $i \notin P$.

These results unify all individual results in this direction under the most succinct conditions. This shows that conditions (1) and (2) are the essence of coexisting in a globally stable positive fixed point, and nearly gives necessary and sufficient conditions for systems with limited competition to coexist in a globally stable positive fixed point. Furthermore, assuming (H1), (1) and (2) hold, we conclude that there is an attractor block lying in the interior of the positive octant with fixed points at two corners, each of which attracts an unbounded open set of positive initial points. By applying Tereščák's result in [18], almost all orbits are asymptotic to periodic points if T is strongly order-preserving in the interior of R_+^n . The convergent result for all orbits is provided under suitable conditions. We also apply our abstract results to periodic ordinary differential equations and reaction-diffusion Lotka-Volterra systems with Neumann boundary conditions:

$$(1.4) \quad \begin{cases} \frac{\partial u}{\partial t} = \text{diag}(d_1, \dots, d_n) \Delta u + \text{diag}(u)(r + Mu) & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega_T, \\ u(x, 0) = u^0(x) & \text{in } \Omega, \end{cases}$$

where M has the form (1.2). As far as we know, most work on (1.4) focusses on two species, and the best results so far were obtained in [19]–[21], where the matrix $M = (a_{ij})_{2 \times 2}$ and the corresponding two-dimensional system (1.3) are assumed to satisfy the hypotheses (H1) through (H3) in all three cases. They proved that there is a steady state attracting all positive initial value functions. Applying our abstract results, we can generalize the best results for two species to any species. Precisely, if $M = (a_{ij})_{n \times n}$ has the form (1.2) and (1.3) satisfies (H1) through (H3), then there is a steady state attracting all $u^0(x) \geq 0$ with $u_i^0(x) \not\equiv 0$ for all i . The necessary and sufficient conditions for (1.4) to coexist in a positive steady state or to exclude in a boundary steady state are also provided.

2. PRELIMINARY

In this section, we will introduce some notation, establish some conventions, and describe some results which are essential tools in the later sections.

Let $R_+^n = \{x \in R^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}$ denote the nonnegative octant, and let $\text{Int}R_+^n = \{x \in R^n : x_i > 0 \text{ for } 1 \leq i \leq n\}$ denote the interior of R_+^n . Define the set $K = \{x \in R^n : x_i \geq 0 \text{ for } 1 \leq i \leq k \text{ and } x_j \leq 0 \text{ for } k+1 \leq j \leq n\}$. We know both R_+^n and K are cones in R^n . We write $x \leq y$ ($x \leq_K y$) and $y \geq x$ ($y \geq_K x$) whenever $y - x \in R_+^n$ ($y - x \in K$), $x < y$ ($x <_K y$) and $y > x$ ($y >_K x$) whenever $x \leq y$ ($x \leq_K y$) and $x \neq y$. If $x, y \in R^n$ and $x \leq y$ ($x \leq_K y$), we let $[x, y] = \{z \in R^n : x \leq z \leq y\}$ ($[x, y]_K = \{z \in R^n : x \leq_K z \leq_K y\}$).

Suppose that A is an $n \times m$ matrix. Then we write $A \geq 0$ if $a_{ij} \geq 0$ for all i and j . If M is an $n \times n$ matrix, and

$$M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix},$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix, C is an $(n-k) \times k$ matrix, D is an $(n-k) \times (n-k)$ matrix and $A, B, C, D \geq 0$, then we write $M \geq_K 0$. It is easy to see that $M(K) \subset K$.

We will reserve the letter n for the dimension of R^n , and $N = \{1, 2, \dots, n\}$. Let L be a subset of N and $\bar{L} = N \setminus L$ be its complementary set in N . We define the sets $H_L^+ = \{x \in R_+^n : x_j = 0 \text{ for } j \in \bar{L}\}$ and $\text{Int}H_L^+ = \{x \in H_L^+ : x_i > 0 \text{ for all } i \in L\}$. In particular, if $L = \emptyset$, then $H_L^+ = \text{Int}H_L^+ = \{(0, 0, \dots, 0)\}$.

A continuous map $S : R_+^n \rightarrow R_+^n$ is said to be *cooperative* if $S(x) < S(y)$ whenever $x < y$ with $x, y \in R_+^n$, and $T : R_+^n \rightarrow R_+^n$ is said to be a map with *limited competition* if $T(x) <_K T(y)$ whenever $x <_K y$ with $x, y \in R_+^n$. In particular, if $k = 0$, then T is also cooperative. We note that a map with limited competition is often said to be type- K *monotone* (see [7], [16]). In this paper we always assume that $S, T \in C^1(R_+^n, R_+^n)$ and S is a cooperative map, and T is a map with limited competition. We also always assume that **each orbit of S or T is bounded**. We denote by S_1, S_2, \dots, S_n and T_1, T_2, \dots, T_n the components of S and T respectively. It is easy to see that $DS = (\frac{\partial S_i}{\partial x_j})_{n \times n} \geq 0$ and $DT = (\frac{\partial T_i}{\partial x_j})_{n \times n} \geq_K 0$.

Now, we give the main hypotheses on T .

(AT) $T(\text{Int}H_L^+) \subset \text{Int}H_L^+$ for any $L \subset N$.

(BT) If x_0 is a fixed point of T with $x_0 \in \text{Int}H_L^+$, then x_0 is locally asymptotically stable with respect to H_L^+ .

(CT) Each fixed point x_0 of T is hyperbolic.

For the cooperative map S , we still suppose that the same hypotheses hold, but use (AS), (BS) and (CS) to replace (AT), (BT) and (CT) respectively.

If a map T satisfies (AT) and a point $x \in H_L^+$, then for any $i \in \bar{L}, j \neq i$, $\frac{\partial T_i}{\partial x_j} \Big|_x = 0$ and $\frac{\partial T_i}{\partial x_i} \Big|_x$ is an eigenvalue of $DT(x)$. A cooperative map S has the similar property if it satisfies (AS).

Suppose $I = \{1, 2, \dots, k\}$ and $J = \{k+1, k+2, \dots, n\}$. Then the restriction of T to H_I^+ or H_J^+ is cooperative. If T satisfies (AT), (BT) and (CT), then the restriction of T to H_I^+ or H_J^+ satisfies (AS), (BS), and (CS).

In sections 3 through 6 we use $\omega(x)$ to denote the omega limit set of x under S or T ; if $\omega(x) = \{y\}$ has only one element, then we write $\omega(x) = y$.

3. GLOBAL STABILITY FOR COOPERATIVE SYSTEMS

The object of this section is to prove the following theorem.

Theorem 3.1. *Suppose that $S : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AS), (BS) and (CS). Then there is a fixed point $p \in R_+^n$ such that $\omega(x) = p$ for any $x \in \text{Int } R_+^n$. Furthermore, a necessary and sufficient condition for S to have a globally asymptotically stable positive fixed point is that there exist a subset $L \subset N$ and a boundary fixed point $c \in \text{Int } H_L^+$ such that $\left. \frac{\partial T_i}{\partial x_i} \right|_c > 1$ for all $i \in \bar{L}$.*

To prove this theorem, we need the following lemma.

Lemma 3.2. *Suppose that $S : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AS), (BS) and (CS), and that there exists a locally asymptotically stable fixed point p in R_+^n . Then $\omega(x) = p$ for any $x \in \text{Int } R_+^n$. Furthermore, let $L = \{i : p_i > 0\}$. Then the domain of attraction of p is $\Omega = \{x \in R_+^n : x_i > 0 \text{ for all } i \in L\}$.*

Proof. Since (AS) implies that H_P^+ is invariant under S for any $P \subset N$, we shall prove this lemma by induction on the dimension n of R_+^n . The result is obvious if $n = 1$. So assume for induction that the lemma is true when the dimension of the system is less than n . Now, we consider an n -dimensional system. First, because p is locally asymptotically stable, there is no other fixed point in $\Omega_1 = p + R_+^n$. In fact, if there is another fixed point in Ω_1 , then we can find a fixed point \bar{p} in Ω_1 such that there is no other fixed point distinct from p and \bar{p} in $[p, \bar{p}]$, because p is locally asymptotically stable and there is no other fixed point in a small neighborhood of p . Assume that $\bar{p} \in \text{Int } H_{L_1}^+$ for some $L_1 \subset N$. Then $[p, \bar{p}] \subset H_{L_1}^+$. By (BS), \bar{p} is locally asymptotically stable with respect to $H_{L_1}^+$. In particular, p and \bar{p} are locally asymptotically stable with respect to $[p, \bar{p}]$. By the result in Dancer and Hess [22], S has a fixed point distinct from p and \bar{p} in $[p, \bar{p}]$, contradicting the conclusion that we have just shown. Because Ω_1 is an invariant set of S , p attracts all points in Ω_1 by the result in Jiang [23]. In the following we divide the proof into two cases.

Case (1): $p \notin \text{Int } R_+^n$. Obviously, there is some $L_2 \neq N$ such that $p \in \text{Int } H_{L_2}^+$. The induction assumption implies that the domain of attraction of p includes the set $\Omega_2 = \{x \in R_+^n : 0 < x_i \leq p_i \text{ for } i \in L_2 \text{ and } x_j = 0 \text{ for } j \in \bar{L}_2\}$. For any $x \in \Omega$, we can find $x^1 \in \Omega_1$ and $x^2 \in \Omega_2$ such that $x^2 \leq x \leq x^1$. Because of the monotonicity of S , $\omega(x^2) = \omega(x) = \omega(x^1) = p$.

Case (2): $p \in \text{Int } R_+^n$. Let $N_i = N \setminus \{i\}$. Denote $x - x_i e_i$ by x_{N_i} for any $x \in R_+^n$, where $x = (x_1, x_2, \dots, x_i, \dots, x_n)$ and $e_i = (0, 0, \dots, 1_i, 0, \dots, 0)$. Obviously, $x_{N_i} \in \text{Int } H_{N_i}^+$ for any $x \in \text{Int } R_+^n$. Since $p_{N_i} < p$, $S(p_{N_i}) < S(p) = p$, i.e., $S_j(p_{N_i}) \leq p_j$ for $j \in N_i$. Hence, $S(p_{N_i}) \leq p_{N_i}$. It is easy to see that there exists a fixed point q_{N_i} such that $\omega(p_{N_i}) = q_{N_i}$. The hypothesis (CS) shows that the local stable set of q_{N_i} is a manifold (with boundary). By (BS), there is $\widetilde{N}_i \subset N_i$ such that $q_{N_i} \in \text{Int } H_{\widetilde{N}_i}^+$ and q_{N_i} is locally asymptotically stable with respect to $H_{\widetilde{N}_i}^+$. On the other hand, q_{N_i} attracts all points in $[q_{N_i}, p_{N_i}]$. So the local stable manifold (with boundary) of q_{N_i} includes a neighborhood of q_{N_i} in $H_{N_i}^+$. The induction assumption shows that the domain of attraction of q_{N_i} includes $\text{Int } H_{N_i}^+$.

Define

$$(3.1) \quad q = (q_1, q_2, \dots, q_n) \text{ with } q_i = \max_{j \in N} (q_{N_j})_i.$$

Then $q_{N_i} \leq q$ for any $i \in N$, and if $q_{N_i} \leq x$ for any $i \in N$, then $q \leq x$. On the other hand, $q_{N_i} = S(q_{N_i}) \leq S(q)$, which implies $q \leq S(q)$. So there exists a fixed point w such that $\omega(q) = w \geq q$.

For any $x \in \text{Int } R_+^n$, $\omega(x) \geq \omega(x_{N_i}) = q_{N_i}$ for any $i \in N$. So $\omega(x) \geq q$; in particular, $\omega(p) = p \geq q$. If $q \in \text{Int } R_+^n$, then $\omega(q) = p$. In fact, $q \leq w = \omega(q) \leq p$. Since $w \in \text{Int } R_+^n$, the result just proved in the first paragraph shows that there is no other fixed point in $w + R_+^n$, and so $w = p$. Because $\omega(x) \geq q$ for any $x \in \text{Int } R_+^n$, we have $\omega(y) \geq \omega(q) = p$ for any $y \in \omega(x)$. Furthermore, $\omega(y) = p$, since Ω_1 is contained in the attracting domain of p . So $\{p\} = \omega(y) \subset \omega(x)$. However, p is locally asymptotically stable; hence $\omega(x) = p$.

If $q \notin \text{Int } R_+^n$, then there is some $i \in N$ such that $q \in H_{N_i}^+$. Since $q \geq q_{N_i}$ and q_{N_i} attracts all points in $\text{Int } H_{N_i}^+$ by the induction assumption, $\omega(q) = w = q_{N_i}$. Because $\omega(q) \geq q \geq q_{N_i}$, we get $q = q_{N_i}$. Since q is locally asymptotically stable with respect to $H_{N_i}^+$, we must have $\left. \frac{\partial S_i}{\partial x_i} \right|_q > 1$. Otherwise, q is locally asymptotically stable in R_+^n . Based on the result in case (1), q attracts all points in $\text{Int } R_+^n$, contradicting the fact that $p \in \text{Int } R_+^n$ is a fixed point.

Now, we show that $q = q_{N_i} \in \text{Int } H_{N_i}^+$. Otherwise, there is some $j \in N_i$ such that $(q_{N_i})_j = 0$. Then $q_{N_i}^\delta = q_{N_i} + \delta e_i \in H_{N_j}^+$ for any positive number δ . Because $\left. \frac{\partial S_i}{\partial x_i} \right|_q > 1$ and $\left. \frac{\partial S_i}{\partial x_j} \right|_q = 0, j \neq i$, we get

$$\begin{aligned} S_i(q_{N_i}^\delta) &= S_i(q_{N_i}^\delta) - S_i(q_{N_i}) \\ &= \left(\int_0^1 DS(sq_{N_i}^\delta + (1-s)q_{N_i}) ds (q_{N_i}^\delta - q_{N_i}) \right)_i \\ &\geq \delta = (q_{N_i}^\delta - q)_i \end{aligned}$$

for sufficiently small δ . This implies that $S(q_{N_i}^\delta) \geq q_{N_i}^\delta > q$. Thus, there is a fixed point \bar{w} such that $\omega(q_{N_i}^\delta) = \bar{w}$ with $\bar{w} > q$. By $q_{N_i}^\delta \in H_{N_j}^+$ and (AS), $S(q_{N_i}^\delta) \in H_{N_j}^+$ and hence $\bar{w} \in H_{N_j}^+$, and $\bar{w} = \omega(\bar{w}) > q \geq q_{N_j}$. On the other hand, since $\bar{w} \in H_{N_j}^+$ and $\bar{w} \geq q_{N_j}$, we have $\omega(\bar{w}) = q_{N_j}$, a contradiction.

By the proof of the previous paragraphs, we know that $q = q_{N_i} \in \text{Int } H_{N_i}^+$ and $\omega(q^\delta) \in \text{Int } R_+^n$ is a fixed point with $q^\delta = q + \delta e_i$ for $0 < \delta < \delta_0$. Furthermore, $\omega(q^\delta) \leq p$. But since $\omega(q^\delta) \in \text{Int } R_+^n$ is asymptotically stable, there is no other fixed point in $\omega(q^\delta) + R_+^n$. So $\omega(q^\delta) = p$. By the induction assumption, q attracts $\text{Int } H_{N_i}^+$. It follows from the hyperbolicity and $\left. \frac{\partial S_i}{\partial x_i} \right|_q > 1$ that the dimension of the stable manifold $W^s(q)$ of q is $n - 1$. Hence $W^s(q) = \text{Int } H_{N_i}^+$. This proves there cannot exist an $x \in \text{Int } R_+^n$ such that $\omega(x) = q$. We claim that $\omega(x) = p$ for all $x \in \text{Int } R_+^n$. Suppose not. Then there exists an $x^0 \in \text{Int } R_+^n$ such that $\omega(x^0) \neq p$. This implies that there is a point $y \in \omega(x^0)$ with $y \neq q, p$. It is easy to see that $\omega(x) \subset [q, p]$ for all $x \in \text{Int } R_+^n$. Therefore, $q < y < p$. Choose $\delta \in (0, \delta_0)$ such that $q^\delta < y < p$. It follows from $\omega(q^\delta) = p$ that $\omega(y) = p$, i.e., $p \in \omega(y) \subset \omega(x^0)$, a contradiction. This completes the proof. \square

Proof of Theorem 3.1. The theorem is true when $n = 1$. Now we proceed by induction on the dimension n of R_+^n . Suppose that the theorem is true when the dimension of the systems is less than n . Then we consider an n -dimensional system

and look for a fixed point p such that it attracts the set $\Omega = \{x \in R_+^n : x_i > 0 \text{ for every } i \text{ with } p_i > 0\}$. By the induction assumption, for any $i \in N$, there is a fixed point $q_{N_i} \in H_{N_i}^+$ with $q_{N_i} \in \text{Int } H_L^+$ for some $L \subset N_i$ such that q_{N_i} attracts all points in $\{x \in H_{N_i}^+ : x_i > 0 \text{ for } i \in L\}$. Suppose that q is the point defined by (3.1). If $q \in \text{Int } R_+^n$, then $S(q) \geq q$ by the proof of Lemma 3.2. So $\omega(q) \in \text{Int } R_+^n$ is a fixed point, and we denote it by p . It follows from (BS) that p is locally asymptotically stable in $\text{Int } R_+^n$. By Lemma 3.2, $\text{Int } R_+^n$ is the domain of attraction of p .

If $q \notin \text{Int } R_+^n$, then the proof of Lemma 3.2 shows that there is an $i \in N$ such that $q = q_{N_i}$ attracts $\text{Int } H_{N_i}^+$. If q is locally asymptotically stable, then, applying Lemma 3.2 to q , we obtain that q attracts the set $\Omega = \{x \in R_+^n : x_j > 0 \text{ for each } j \text{ with } q_j > 0\}$. Therefore, q is the desired fixed point. Otherwise $\left. \frac{\partial S_i}{\partial x_i} \right|_{q_{N_i}} > 1$, and hence, by

the proof of Lemma 3.2, $q_{N_i} \in \text{Int } H_{N_i}^+$, and $q_{N_i}^\delta = q_{N_i} + \delta e_i \in \text{Int } R_+^n$, $S(q_{N_i}^\delta) \geq q_{N_i}^\delta$. Hence $\omega(q_{N_i}^\delta) \in \text{Int } R_+^n$ is a fixed point. If we denote the positive fixed point $\omega(q_{N_i}^\delta)$ by p , then p is locally asymptotically stable by (BS). Lemma 3.2 implies p attracts all points in $\text{Int } R_+^n$. Such a fixed point p is a desired one.

In the following, we shall prove the remaining part of the theorem. Suppose that S has a globally asymptotically stable positive fixed point p . Then the fixed point 0 is not asymptotically stable. Indeed, if 0 is asymptotically stable, then 0 is globally asymptotically stable by Lemma 3.2, a contradiction. Without loss of generality, suppose $\left. \frac{\partial S_1}{\partial x_1} \right|_0 > 1$. Then, it is easy to see that $S(\epsilon e_1) > \epsilon e_1$ for all sufficiently small positive numbers ϵ . So $a = \omega(\epsilon e_1) \in \text{Int } H_{\{1\}}^+$ is a fixed point with $0 < a < p$. Similarly, a is not asymptotically stable. Suppose $\left. \frac{\partial S_2}{\partial x_2} \right|_a > 1$. Then $S(a + \epsilon e_2) > a + \epsilon e_2$ for all sufficiently small positive numbers ϵ . There is a fixed point $b = \omega(a + \epsilon e_2) \in \text{Int } H_{\{1,2\}}^+$ with $a < b < p$ that is not asymptotically stable. In this way, we can find a fixed point $c \in \text{Int } H_{N_i}^+$ such that c is not asymptotically stable. This means $\left. \frac{\partial S_i}{\partial x_i} \right|_c > 1$. The proof of necessity is complete.

On the other hand, suppose there exist a subset $L \subset N$ and a boundary fixed point $c \in \text{Int } H_L^+$ such that $\left. \frac{\partial S_i}{\partial x_i} \right|_c > 0$ for all $i \in \bar{L}$. Without loss of generality, suppose $L = \{1, \dots, l\}$, $c = (c_1, \dots, c_l, 0, \dots, 0)$ and $\bar{c} = (c_1, \dots, c_l, \epsilon, \dots, \epsilon)$, where ϵ is a sufficiently small positive number. Then $\bar{c} \in \text{Int } R_+^n$ and $S(\bar{c}) > \bar{c}$. Therefore, $p = \omega(\bar{c}) \in \text{Int } R_+^n$ is a positive fixed point that is asymptotically stable by (BS). Furthermore, it follows from Lemma 3.2 that p is globally asymptotically stable. The proof is complete. \square

Remark 3.3. For autonomous ordinary differential equations, similar results were proved in [25, 26]. Theorem 3.1 can be viewed as a unified generalization of such individual results.

4. GLOBAL STABILITY FOR SYSTEMS WITH LIMITED COMPETITION

The object of this section is to prove the following theorem, which is about the global stability of systems with limited competition.

Theorem 4.1. *Suppose that $T : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AT), (BT) and (CT). Then there is a fixed point $p \in R_+^n$ such that $\omega(x) = p$ for any $x \in \text{Int } R_+^n$. Furthermore, let $L = \{i : p_i > 0\}$. Then the domain of attraction of p is $\Omega = \{x \in R_+^n : x_i > 0 \text{ for all } i \in L\}$.*

To prove this theorem, we first give a lemma about the index of fixed points.

Lemma 4.2. *Suppose that $T : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AT), (BT) and (CT). Let $U = R_+^n$ or $U = H_L^+$ or $U = [p, q]_K$, where p and q are fixed points of T . Then for any fixed point $x^0 \in U$, either $i(T, U, x^0) = 0$ or $i(T, U, x^0) = 1$. Furthermore, $i(T, U, x^0) = 0 \Leftrightarrow DT|_{x^0}$ has at least one eigenvalue with modulus larger than 1.*

Proof. Obviously, U is an invariant set under T . If there exists $L \neq N$ such that $U \subset H_L^+$, then the restriction of T to H_L^+ also satisfies (AT), (BT) and (CT). So we only need to consider the case that $U \cap \text{Int } R_+^n \neq \emptyset$.

Suppose that $x^0 \in U$ is a fixed point. Then

$$DT(x^0)(x - x^0) = \lim_{\tau \rightarrow 0^+} \tau^{-1}(T(\tau(x - x^0) + x^0) - T(x^0)) \text{ for any } x \in U.$$

Let $P = x^0 + \bigcup_{t=0}^{+\infty} t(U - x^0)$. Then it is easy to see that P is a closed set. For any $x \in U$ and $0 \leq \tau \leq 1$, $\tau(x - x^0) + x^0 \in U$. Hence $T(\tau(x - x^0) + x^0) \in U$. So $\tau^{-1}(T(\tau(x - x^0) + x^0) - T(x^0)) + x^0 \in P$. This implies that $DT(x^0)(x - x^0) + x^0 \in P$.

Suppose that $B(x^0, \delta)$ is a closed ball in R_+^n whose center is x^0 and radius is δ , where δ is sufficiently small so that $B^+(x^0, \delta) = B(x^0, \delta) \cap P \subset U$. Because $DT(x^0)$ is a linear map, there is an ϵ such that $DT(x^0)(x - x^0) + x^0 \in B^+(x^0, \delta) \subset U$ for any $x \in B^+(x^0, \epsilon)$.

Since $DT(x^0)$ is hyperbolic, there is $\alpha > 0$ such that

$$\|x - (x^0 + DT(x^0)(x - x^0))\| \geq \alpha \|x - x^0\|.$$

Choose $\sigma < \epsilon$ such that $\|T(x) - T(x^0) - DT(x^0)(x - x^0)\| \leq \frac{\alpha}{2} \|x - x^0\|$ for any $x \in B^+(x^0, \sigma)$. For any y with $\|y\| < \sigma\alpha/2$ and $x^0 + y \in U$, and any $x \in U$ with $\|x - x^0\| = \sigma$,

$$\begin{aligned} & \|x - (1 - \lambda)(DT(x^0)(x - x^0) + x^0 + y) - \lambda T(x)\| \\ &= \|(x - DT(x^0)(x - x^0) - x^0) + \lambda(DT(x^0)(x - x^0) + x^0 - T(x)) - (1 - \lambda)y\| \\ &\geq \|x - DT(x^0)(x - x^0) - x^0\| - \lambda\|DT(x^0)(x - x^0) + x^0 - T(x)\| - (1 - \lambda)\|y\| \\ &> \alpha\sigma - \alpha\sigma/2 - \alpha\sigma/2 = 0. \end{aligned}$$

For any $\lambda \in [0, 1]$, the homotopy invariance of the fixed point index (e.g., [24]) implies $i(T, U, x^0) = i(DT(x^0)(x - x^0) + x^0 + y, U, x^0)$.

(1) Assume that $DT(x^0)$ has no eigenvalue whose modulus is larger than 1. Then choose $y = 0$. The homotopy invariance implies

$$\begin{aligned} & i(T, U, x^0) \\ &= i(DT(x^0)(x - x^0) + x^0, U, x^0) \\ &= i(\lambda DT(x^0)(x - x^0) + x^0, U, x^0) \quad \lambda \in [0, 1] \\ &= i(x^0, U, x^0) = 1. \end{aligned}$$

(2) $DT(x^0)$ has an eigenvalue whose modulus is larger than 1. The hypothesis (BT) implies $x^0 \notin \text{Int } R_+^n$; hence there is $L \neq N$ such that $x^0 \in \text{Int } H_L^+$. Without loss of generality, suppose $x^0 = (x_L^0, 0)$, where $x_L^0 \in \text{Int } R_+^l$ with $l = \#L$. Then $DT(x^0)$ has the following form:

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where A is an $l \times l$ matrix, all of whose eigenvalues have moduli less than 1, and $D = \text{diag}(\frac{\partial T_i}{\partial x_j})|_{x^0}$, $j \in \bar{L}$. It is easy to see if λ is one eigenvalue of $DT(x^0)$ whose modulus is larger than 1, then there is some $i \in \bar{L}$ such that $\lambda = \frac{\partial T_i}{\partial x_i}|_{x^0}$.

On the other hand, $U = [p, q]_K$ can be rewritten as the form $[(p_I, q_J), (q_I, p_J)]$, where (p_I, q_J) and (q_I, p_J) are elements of R_+^n , $p - (p_I, q_J) \in H_J^+$, $q - (p_I, q_J) \in H_I^+$, $(q_I, p_J) - p \in H_I^+$, $(q_I, p_J) - q \in H_J^+$. Because $U \cap \text{Int}R_+^n \neq \emptyset$, $(q_I, p_J) \in \text{Int}R_+^n$. Since $x^0 \in U$ and $(q_I, p_J) \in \text{Int}R_+^n$, there is a sufficiently small a such that $x^0 + ae_i \in U$ for $i \in \bar{L}$. We shall show that $x - (DT(x^0)(x - x^0) + x^0) = \beta e_i$ has no solution in $B^+(x^0, \sigma)$ for any $\beta > 0$. In fact, $x_i^0 = 0, x_i \geq 0$. Let A_i be the i -th component of $x - (DT(x^0)(x - x^0) + x^0)$. Then

$$A_i = (1 - \partial T_i / \partial x_i|_{x^0})(x_i - x_i^0) = (1 - \lambda)(x_i - x_i^0) \leq 0,$$

but the i -th component of βe_i is larger than 0.

Let $y = \beta e_i$, where β is sufficiently small so that $\|y\| < \sigma\alpha/2$. Then

$$i(T, U, x^0) = i(DT(x^0)(x - x^0) + x^0 + y, U, x^0).$$

By the above proof, $DT(x^0)(x - x^0) + x^0 + y$ has no fixed point in $B^+(x^0, \sigma)$, which implies $i(DT(x^0)(x - x^0) + x^0 + y, U, x^0) = 0$. So $i(T, U, x^0) = 0$. \square

In the following, we will show if there exists a fixed point that is locally asymptotically stable, then it is globally asymptotically stable.

Lemma 4.3. *Suppose that $T : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AT), (BT) and (CT). Assume that there exists a fixed point a such that it attracts $\text{Int}H_L^+$ with $L \supset J$, $a_i > 0$ for $i \in L \cap I$, and $\frac{\partial T_l}{\partial x_l}|_a > 1$ for some $l \in I \setminus L$. Then there is a fixed point $b \in H_{L_1}^+$ with $L_1 = L \cup \{l\}$ such that*

- (i) $b_i > 0$ for any $i \in I \cap L_1$,
- (ii) b attracts the set

$$\Gamma = \{x \in H_{L_1}^+ : 0 < x_i < b_i \text{ for } i \in I \cap L_1, x_{i'} = 0 \text{ for } i' \in I \setminus L_1, \\ \text{and } x_j > b_j \text{ for any } j \in J\},$$

- (iii) if the fixed point $p \geq_K a$ with $p_l > 0$, then $b \leq_K p$.

Furthermore, the same conclusion holds if I and J are switched.

Proof. To be specific, we assume $L = \{m, m+1, \dots, n\}$ with $m \leq k+1$. Since a attracts $\text{Int}H_L^+$, we have $a = (0, \dots, 0, a_m, a_{m+1}, \dots, a_n)$ with $a_i > 0$ for $m \leq i \leq k$. By $\frac{\partial T_l}{\partial x_l}|_a > 1$, we have $l < m$ and $T_l(a + \delta e_l) > \delta$ for $0 < \delta \ll 1$. Let $\bar{a} = a + \delta e_l$. Then the monotonicity implies that $T(\bar{a}) >_K \bar{a}$. Hence $\omega(\bar{a}) = b >_K \bar{a}$. It is easy to see that b satisfies (i) and is independent of δ for $0 < \delta \ll 1$.

Set $x \in \Gamma$. Then $x - x_l e_l <_K x <_K b$. Obviously, $x - x_l e_l \in \text{Int}H_L^+$. Therefore, $a = \omega(x - x_l e_l) \leq_K \omega(x) \leq_K b$ by the assumption and monotonicity, that is, $\omega(x) \subset [a, b]_K$ for any $x \in \Gamma$.

Because T is continuously differentiable and $\frac{\partial T_l}{\partial x_l}|_a > 1$, there is a point $\hat{a} = (0, \dots, 0, \hat{a}_m, \dots, \hat{a}_k, \dots, \hat{a}_n)$ with $0 < \hat{a}_i < a_i$ for $m \leq i \leq k$ and $\hat{a}_j > a_j$ for any $j \in J$ such that $\frac{\partial T_l}{\partial x_l}|_{\hat{a}} > 1$, which implies that there is a $\delta_0 > 0$ such that $T_l(\hat{a} + \epsilon e_l) \geq \epsilon$ for $0 \leq \epsilon \leq \delta_0$.

From now on we fix $x \in \Gamma$. From $\omega(x) \subset [a, b]_K$ it follows that there is an integer R such that $T^r(x) \in [\hat{a}, b]_K$ for all $r \geq R$. By (AT), $T_l(T^r(x)) > 0$ for any r . We claim that there is a point $y \in \omega(x)$ with $y_l > 0$. Otherwise, there is

$\bar{R} \geq R$ such that $(T^r(x))_l < \delta_0$ for any $r \geq \bar{R}$. Obviously, $T^r(x) \geq_K \hat{a} + (T^r(x))_l e_l$. The monotonicity implies that $(T^{r+1}(x))_l \geq T_l(\hat{a} + (T^r(x))_l e_l) \geq (T^r(x))_l$ for all $r \geq \bar{R}$. Hence $y_l > 0$ for all $y \in \omega(x)$, a contradiction. The claim holds. Fix $y \in \omega(x)$ with $y_l > 0$. Then $a \leq_K y$, and hence we can choose $0 < \delta \ll 1$ such that $\bar{a} = a + \delta e_l \leq_K y \leq_K b$. This shows that $\omega(y) = \omega(\bar{a}) = b$, i.e., $b \in \omega(x)$. Since for any $x \in \Gamma$ we have $x <_K b$ and $x_i \neq b_i$ for all $i \in L_1$, we can find an integer r such that $x <_K T^r(x)$, which implies that $\omega(x)$ is a periodic orbit. Together with the fixed point $b \in \omega(x)$, we conclude that $\omega(x) = b$ for all $x \in \Gamma$.

Finally, if p is a fixed point with $a \leq_K p$ and $p_l > 0$, then $\bar{a} = a + \delta e_l \leq_K p$ for $0 < \delta \ll 1$, which implies that $b = \omega(\bar{a}) \leq_K p$. The proof is complete. \square

Lemma 4.4. *Suppose that $T : R_+^n \rightarrow R_+^n$ satisfies the hypotheses (AT), (BT) and (CT), and that there exists a locally asymptotically stable fixed point p in R_+^n . Then $\omega(x) = p$ for any $x \in \text{Int } R_+^n$. Furthermore, let $L = \{i : p_i > 0\}$. Then the domain of attraction of p is $\Omega = \{x \in R_+^n : x_i > 0 \text{ for all } i \in L\}$.*

We shall prove this lemma by induction on the dimension n of R_+^n .

Lemma 4.5. *When $n = 2$, the conclusion of Lemma 4.4 holds.*

Proof. In this case, $N = \{1, 2\}$, $I = \{1\}$, $J = \{2\}$. The restriction of T to H_I^+ or H_J^+ is cooperative. By Theorem 3.1, there are $(x_I^0, 0) \in H_I^+$ and $(0, x_J^0) \in H_J^+$ such that $\omega(x) = (x_I^0, 0)$ for all $x \in \text{Int } H_I^+$ and $\omega(x) = (0, x_J^0)$ for all $x \in \text{Int } H_J^+$. For any $x = (x_1, x_2) \in \text{Int } R_+^2$, we have $(0, x_2) <_K x <_K (x_1, 0)$. Hence $\omega(x) \subset U = [(0, x_J^0), (x_I^0, 0)]_K$, and $p \in [(0, x_J), (x_I, 0)]_K$.

(1) $p = (0, 0)$. Then $p = (0, x_J^0) = (x_I^0, 0)$, and hence $\omega(x) = p$ for any $x \in R_+^2$.

(2) $p \in \text{Int } H_I^+$. Then $p = (x_I^0, 0)$. Consider $\Omega_1 = \{x \in R_+^2, p \leq_K x\} \subset \text{Int } H_I^+$. Then $\omega(x) = p$ for every $x \in \Omega_1$. Let $\Omega_2 = \{x = (x_1, x_2) \in R_+^2 : 0 < x_1 \leq x_I^0, x_2 > 0\}$. Lemma 4.2 implies that $i(T, U, p) = 1$, and $i(T, U, U) = 1$. Hence if $q \neq p$ is a fixed point in U , then $i(T, U, q) = 0$. So there is no other fixed point in U except $p, (0, x_J^0), (0, 0)$. In fact x_J^0 may be equal to 0.

By Lemma 4.2, $i(T, U, (0, x_J^0)) = 0$ implies $\frac{\partial T_1}{\partial x_1}|_{(0, x_J^0)} > 1$. This condition implies that $T_1((\delta, x_J^0)) > \delta$ for sufficiently small δ . On the other hand, $T((\delta, x_J^0)) >_K T((0, x_J^0)) = (0, x_J^0)$; hence $T((\delta, x_J^0)) >_K (\delta, x_J^0)$. So $\omega((\delta, x_J^0)) = p$; furthermore, by the monotonicity of T , $\omega(x) = p$ for any $x \in U \setminus H_J^+$. There exists $\bar{x}_J^0 > x_J^0$ such that $\frac{\partial T_1}{\partial x_1}|_{(0, \bar{x}_J^0)} > 1$. It follows that $\omega(x) \cap H_J^+ = \emptyset$ for any $x \in \text{Int } R_+^2$, and we know that $\omega(x) \subset U = [(0, x_J^0), (x_I^0, 0)]_K$. Hence $\omega(x) = p$.

(3) $p \in \text{Int } H_J^+$. This case is similar to (2).

(4) $p \in \text{Int } R_+^2$. In this case, it is easy to see that there exist precisely four fixed points $(0, 0), (x_I^0, 0), (0, x_J^0)$ and p in R_+^2 , and $(0, x_J^0) <_K p, (0, 0) <_K (x_I^0, 0)$. $p \in \text{Int } R_+^2$ implies p is locally asymptotically stable, and $i(T, U, p) = 1$. Hence the other fixed points have index 0. In the same way as in (2), we can obtain that $\omega(x) \cap H_I^+ = \omega(x) \cap H_J^+ = \emptyset$ for any $x \in \text{Int } R_+^2$, and $\omega(x) = p$ for any $x \in U \cap \text{Int } R_+^2$. Hence $\omega(x) = p$ for any $x \in \text{Int } R_+^2$. \square

Proof of Lemma 4.4. We shall prove this lemma by induction on the dimension n of R_+^n . The statement is clear if $n = 2$ by Lemma 4.5. So assume for induction that the lemma is true when the dimension of the system is less than n . Now, we consider an n -dimensional system. We divide the proof into three cases.

- (1) $p \in H_I^+$, (2) $p \in H_J^+$, (3) $p \notin H_I^+$ and $p \notin H_J^+$.

Case (1). Let $p = (p_1, p_2, \dots, p_k, 0, 0, \dots, 0)$. Then p is locally asymptotically stable by assumption. In particular, p is locally asymptotically stable with respect to H_I^+ . Because of the conclusion of Lemma 3.2, p attracts all points in $\{x \in H_I^+ : x_i \neq 0 \text{ for any } i \text{ with } p_i \neq 0\}$. In particular, p attracts all points in $\Omega_1 = \{x \in R_+^n, x \geq_K p\}$. In the following, we consider the set Ω_2 defined by

$$\Omega_2 = \{x \in R_+^n : 0 < x_i < p_i \text{ for } i \in I \text{ with } p_i > 0, \\ x_{i'} = 0 \text{ for } i' \in I \text{ with } p_{i'} = 0, x_j > 0 \text{ for } j \in J\}.$$

If there exists $i \in I$ such that $p_i = 0$, then $\Omega_2 \subset H_{N_i}^+$ where $N_i = N \setminus \{i\}$, and p is locally asymptotically stable with respect to $H_{N_i}^+$. Hence, p attracts all the points in Ω_2 by the induction assumption. It is not difficult to prove that for any $x \in \Omega$, there exist $y \in \Omega_2$ and $z \in \Omega_1$ such that $y \leq_K x \leq_K z$, which implies that $\omega(x) = \omega(y) = \omega(z) = p$. The lemma is true in this situation.

Now, suppose that $p \in \text{Int}H_I^+$. H_J^+ is an invariant set under T , and T is cooperative on H_J^+ . Hence there is a fixed point $a = (0, \dots, 0, a_{k+1}, \dots, a_n) \in H_J^+$ such that a attracts all points in

$$\{x \in H_J^+ : x_j > 0 \text{ for any } j \in J \text{ with } a_j > 0\}.$$

$U = [a, p]_K$ is invariant under T . Since p is locally asymptotically stable and $i(T, U, U) = 1$, Lemma 4.2 implies that $i(T, U, p) = 1$ and $i(T, U, q) = 0$ for any other fixed point $q \in U$. Since $i(T, U, a) = 0$, by Lemma 4.2 there is $i \in I$ such that $\frac{\partial T_i}{\partial x_i}|_a > 1$ is an eigenvalue of $DT(a)$. Without loss of generality, let $i = 1$. Then $a, L = J$ and $l = 1$ satisfy all conditions in Lemma 4.3. Applying it, we have a fixed point a^1 satisfying (i) through (iii) with $L_1 = J \cup \{1\}$. In particular, $a^1 \in [a, p]_K$ attracts $\{x \in H_{L_1}^+ : 0 < x_1 < a_1^1, x_j > a_j^1 \text{ for } j \in J\}$. By the hyperbolic assumption, a^1 is locally asymptotically stable in $H_{L_1}^+$. Therefore if $I \neq \{1\}$, then a^1 attracts $\text{Int}H_{L_1}^+$ by the induction assumption. Applying Lemma 4.2 again, we get that there is $i \in I \setminus \{1\}$, say $i = 2$, such that $\frac{\partial T_2}{\partial x_2}|_{a^1} > 1$. Again, a^1, L_1 and $l = 2$ satisfy all conditions of Lemma 4.3. It follows that there exists a fixed point $a^2 \in [a, p]_K$ such that it attracts the set $\{x \in H_{L_2}^+ : 0 < x_i < a_i^2 \text{ for } i = 1, 2, x_j > a_j^2 \text{ for } j \in J\}$ with $L_2 = L_1 \cup \{2\}$. The hyperbolic assumption implies that a^2 is locally asymptotically stable in $H_{L_2}^+$. If $I \neq \{1, 2\}$, we conclude that a^2 attracts $\text{Int}H_{L_2}^+$ by the induction assumption.

After using such a procedure k times, we can find a fixed point $a^k = q = (q_1, q_2, \dots, q_k, q_{k+1}, \dots, q_n) \in [a, p]_K$ with $q_i > 0$ for $i \in I$. The domain of attraction of q includes

$$\Omega_q = \{x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) : 0 < x_i < q_i \text{ for } i \in I, x_j > q_j \text{ for any } j \in J\}.$$

Since q is hyperbolic, it is easy to see that q is locally asymptotically stable with respect to R_+^n ; in particular, q is locally asymptotically stable with respect to $[a, p]_K$. Hence, q is equal to p by the uniqueness of the stable fixed point in $[a, p]_K$. Hence, $\Omega_q = \Omega_2$.

For any $x \in \Omega$, there are $x^1 \in \Omega_1$ and $x^2 \in \Omega_2$ such that $x^2 \leq_K x \leq_K x^1$. Therefore, $\omega(x) = \omega(x^1) = \omega(x^2) = p$.

Case (2). If Case (1) is true, then pick a transformation by (x_1, \dots, x_n) such that it is changed into Case (1). Actually, we only choose a suitable permutation of species indices for this purpose.

Case (3). Let $p = (p_1, \dots, p_k, p_{k+1}, \dots, p_n) = (p_I, p_J)$. First, consider the set

$$\begin{aligned} \Omega_3 = \{ & x \in R_+^n : 0 < x_i < p_i \text{ for } i \in I \text{ with } p_i > 0, \\ & x_{i'} = 0 \text{ for } i' \in I \text{ with } p_{i'} = 0, x_j > p_j \text{ for } j \in J\}. \end{aligned}$$

If there is some $i \in I$ such that $p_i = 0$, then there is some $L \subset N$ such that $\Omega_3 \subset H_L^+$. Hence the induction assumption implies $\omega(x) = p$ for any $x \in \Omega_3$.

Otherwise, $p_i > 0$ for any $i \in I$. By Theorem 3.1, there is a fixed point $a \in H_J^+$ that is globally asymptotically stable with respect to $\text{Int}H_J^+$. Since a attracts $\text{Int}H_J^+$, $a \leq_K p$. By Lemma 4.2 and the stability of p , $i(T, [a, p]_K, p) = 1$ and the index of every other fixed point in $[a, p]_K$ is 0. Hence there is $i \in I$ such that $\frac{\partial T_i}{\partial x_i}|_a > 1$ is an eigenvalue of $DT(a)$. Without loss of generality, let $i = 1$; then, applying Lemma 4.3 to a , $L = J$ and $l = 1$ here, we conclude that there is a fixed point $a^1 \in [a, p]_K$ with the property that it attracts the set $\{x \in H_{J \cup \{1\}}^+ : 0 < x_1 < a_1^1, x_j > a_j^1 \text{ for any } j \in J\}$. Completely repeating the argument in Case (1), we obtain that after k steps we can find $q \in [a, p]_K$ that attracts

$$\begin{aligned} \Omega_3^q = \{ & x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) : \\ & 0 < x_i < q_i \text{ for } i \in I, x_j > q_j \text{ for any } j \in J\}. \end{aligned}$$

This implies that q is locally asymptotically stable with respect to R_+^n by hyperbolicity. Therefore, $q = p$ by the uniqueness of the stable fixed point in $[a, p]_K$; that is, p attracts Ω_3 .

Similarly, we can obtain that p attracts all points in

$$\begin{aligned} \Omega_4 = \{ & x \in R_+^n : x_i > p_i \text{ for } i \in I, 0 < x_j < p_j \text{ for } \\ & j \in J \text{ with } p_j > 0, x_{j'} = 0 \text{ for } j' \in J \text{ with } p_{j'} = 0\}. \end{aligned}$$

For any $x \in \Omega$, we can find $x^3 \in \Omega_3$ and $x^4 \in \Omega_4$ such that $x^3 \leq_K x \leq_K x^4$. Then $\omega(x) = \omega(x^3) = \omega(x^4) = p$. The proof is complete. \square

Proof of Theorem 4.1. First, Theorem 3.1 implies that there is a fixed point $a = (0, \dots, 0, a_{k+1}, \dots, a_n) \in H_J^+$ such that a attracts all points in $\{x \in H_J^+ : x_j > 0 \text{ for any } j \in J \text{ such that } a_j > 0\}$. If a is locally asymptotically stable with respect to R_+^n , then by Lemma 4.4, $p = a$ is the point we want to obtain. Otherwise, there is $i \in I$ such that $\frac{\partial T_i}{\partial x_i}|_a > 1$ is an eigenvalue of $DT(a)$. Without loss of generality, we may assume $i = 1$. Applying Lemma 4.3 to a , $L = J$ and $l = 1$ here, we obtain a fixed point $a^1 >_K a$. By (ii) of Lemma 4.3 and Lemma 4.4, a^1 attracts $\text{Int}H_{L_1}^+$ with $L_1 = J \cup \{1\}$. If a^1 is locally asymptotically stable, then $p = a^1$ is the point we want to find. Otherwise we discuss the system in a similar way. At most after the k -th step we can find a fixed point p that is locally asymptotically stable. Then Lemma 4.4 implies the domain of attraction of p is Ω . The proof is complete. \square

Remark 4.6. Similar results for autonomous ordinary differential equations were verified in [7, 15, 16, 17]. Theorem 4.1 unifies quite a number of such individual results.

5. CONVERGENCE FOR SYSTEMS WITH LIMITED COMPETITION

In this section, we will provide a result on the convergence of all orbits for dynamical systems with limited competition.

We say this kind of map T is *strong* in $\text{Int } R_+^n$ if $T(x) \ll_K T(y)$ whenever $x <_K y$ and $x, y \in \text{Int } R_+^n$, where $x \ll_K y$ is understood to be $x_i < y_i$ for any $i \in I$ and $x_j > y_j$ for any $j \in J$.

Theorem 5.1. *Suppose that T is strong in $\text{Int } R_+^n$ and satisfies the hypotheses (AT) with the following additional conditions:*

- (i) *every boundary fixed point is hyperbolic in R_+^n and asymptotically stable relative to the face it belongs to;*
- (ii) *every positive fixed point is stable.*

Then every orbit converges to a fixed point. Furthermore, if a positive fixed point exists, then the positive fixed point set F is either a singleton or a totally ordered curve such that $\omega(x) \subset F$ for any $x \in \text{Int } R_+^n$.

Proof. Assume that there is no positive fixed point. Then the conclusion follows immediately from Theorem 4.1. Again applying Theorem 4.1, we know that $\omega(x)$ is a single fixed point if $x \in \partial R_+^n$. It remains to consider the case that there is a positive fixed point (say $p = (p_1, p_2, \dots, p_n) \in \text{Int } R_+^n$) and $x \in \text{Int } R_+^n$.

Suppose $p_{N_i} = p - p_i e_i \in \text{Int } H_{N_i}^+$ for $1 \leq i \leq k$. Then Theorem 4.1 implies that $\omega(p_{N_i}) = q_{N_i} \in H_{N_i}^+$. Obviously, $p_{N_i} <_K p$ for any $i \in I$. By monotonicity, $T(p_{N_i}) <_K p$. Using (AT), we get that $T(p_{N_i}) \leq_K p_{N_i}$ for any $i \in I$. Therefore, we have $q_{N_i} = \omega(p_{N_i}) \leq_K p_{N_i}$; in particular, $(q_{N_i})_j \geq (p_{N_i})_j = p_j > 0$ for all $i \in I$ and $j \in J$. Let

$$q = (q_1, \dots, q_k, q_{k+1}, \dots, q_n) \text{ with } q_m = \begin{cases} \max_{i \in I} ((q_{N_i})_m) & \text{if } m \in I, \\ \min_{i \in I} ((q_{N_i})_m) & \text{if } m \in J. \end{cases}$$

Then $q_{N_i} \leq_K q \leq_K p$ for any $i \in I$. Moreover, q has the property that if $q_{N_i} \leq_K x$ for any $i \in I$, then $q \leq_K x$. The monotonicity implies that $q_{N_i} = T(q_{N_i}) \leq_K T(q)$ for any $i \in I$; hence $q \leq_K T(q)$. In the following we shall consider the order interval $[q, p]_K$. We divide the cases into $q \in \text{Int } R_+^n$ and $q \notin \text{Int } R_+^n$.

Assume that $q \in \text{Int } R_+^n$. Then $F \cap [q, p]_K$ is either $\{p\}$ or a totally ordered curve by the result of Dancer and Hess in [22], and $\omega(x)$ is a fixed point for any $x \in [q, p]_K$. Fix $x \in \text{Int } R_+^n$ with $x \leq_K p$. Then $x_{N_i} = x - x_i e_i \leq_K x \leq_K p$. Therefore $q_{N_i} = \omega(x_{N_i}) \leq_K \omega(x) \leq_K p$ for each $i \in I$; that is, $\omega(x) \subset [q, p]_K$ for any positive point $x \leq_K p$. By the invariance and the stability of positive fixed points, $\omega(x)$ is a singleton in $F \cap [q, p]_K$.

Assume that $q \notin \text{Int } R_+^n$. Then there exists an $i \in I$ such that $q \in H_{N_i}^+$. Since $q_{N_i} \leq_K q \leq_K p$ and $q_i = 0$, it follows that $q_{N_i} \leq_K q \leq_K p_{N_i}$, which implies that $\omega(q) = q_{N_i}$. It follows from $q \leq_K T(q)$ that $\omega(q) = q_{N_i} = q$. It is easy to see that all fixed points in $[q, p]_K \setminus \{q\}$ are in $\text{Int } R_+^n$. We claim that $\frac{\partial T_i}{\partial x_i}|_q > 1$. Otherwise, q is asymptotically stable. Then there is no other fixed point in a small neighborhood of q . This implies that there is a minimal element q' in the set $F \cap ([q, p]_K \setminus \{q\})$. So the result of Dancer and Hess [22] implies that there is a connecting orbit between q and q' . However, q and q' are stable, a contradiction, which shows that the claim holds.

Using the same method as in the proof of Lemma 4.3, we can easily verify that there is a fixed point $\bar{x} \in \text{Int } R_+^n$ such that $q \leq_K \bar{x} \leq_K p$. Furthermore, either $\bar{x} = p$ or $[\bar{x}, p]_K \cap F$ is a totally ordered curve and $\omega(x)$ is a singleton in $[\bar{x}, p]_K$ for any $x \in \text{Int } R_+^n$ with $x \leq_K p$. Switching the index subsets I and J , we can similarly

prove that $\omega(x)$ is a singleton for any $x \in \text{Int } R_+^n$ with $p \leq_K x$. This completes the proof. \square

Remark 5.2. Theorem 5.1 is a version of Dancer and Hess [22] and Takáč [29]. However, owing to the invariance of the boundary of R_+^n , there are many unstable fixed points on ∂R_+^n .

6. PERMANENCE AND COEXISTENCE IN A POSITIVE FIXED POINT

In this section, we will study permanence for systems with limited competition and use the result in section 4 to give necessary and sufficient conditions for a system with limited competition to have a globally asymptotically stable positive fixed point.

Let L and P be two subsets of N , $I \subset L, J \subset P$, $x = (x_I, x_J) \in R_+^n, x_I \in R_+^k, x_J \in R_+^{n-k}$.

In this section, we always use u, \bar{u}, u^0 , etc. to denote vectors in $\#L$ -dimensional Euclidean space $R^{\#L}$, use v, \bar{v}, v^0 , etc. to denote vectors in $\#\bar{L}$ -dimensional Euclidean space $R^{\#\bar{L}}$, use w, \bar{w}, w^0 , etc. to denote vectors in $\#P$ -dimensional Euclidean space $R^{\#P}$, and use z, \bar{z}, z^0 , etc. to denote vectors in $\#\bar{P}$ -dimensional Euclidean space $R^{\#\bar{P}}$. Without loss of generality, let $L = \{1, 2, \dots, k, k+1, \dots, l\}$, $P = \{n-p+1, \dots, k+1, \dots, n\}$.

We have the following theorem.

Theorem 6.1. *Suppose that T satisfies the hypothesis (AT), $(u^0, 0)$ and $(0, w^0)$ are two fixed points with $(u^0, 0) \in \text{Int } H_L^+$, $(0, w^0) \in \text{Int } H_P^+$ and $(0, w^0) \leq_K (u^0, 0)$. Suppose that $\frac{\partial T_i}{\partial x_i}|_{(u^0, 0)} > 1$ for any $i \in \bar{L}$ and $\frac{\partial T_j}{\partial x_j}|_{(0, w^0)} > 1$ for any $j \in \bar{P}$. Then there are positive fixed points \bar{x}, \hat{x} satisfying the following properties:*

- (1) $\bar{x}, \hat{x} \in \text{Int } R_+^n$, $\bar{x}, \hat{x} \in [(0, w^0), (u^0, 0)]_K$, $\bar{x} \leq_K \hat{x}$.
- (2) If $x \in \text{Int } R_+^n$, $(0, w^0) \leq_K x \leq_K \bar{x}$, then $\omega(x) = \bar{x}$, and if $x \in \text{Int } R_+^n$, $\hat{x} \leq_K x \leq_K (u^0, 0)$, then $\omega(x) = \hat{x}$.
- (3) If $x \in \text{Int } R_+^n$, $(0, w^0) \leq_K x \leq_K (u^0, 0)$, then $\omega(x) \subset [\bar{x}, \hat{x}]_K$.

In addition, if $(u^0, 0)$ attracts all points $x \in \text{Int } H_L^+$ with $x \geq_K (u^0, 0)$, and $(0, w^0)$ attracts all points $x \in \text{Int } H_P^+$ with $x \leq_K (0, w^0)$, then $\omega(x) = \hat{x}$ for all $x \in \text{Int } R_+^n$ with $x \geq_K \hat{x}$, $\omega(x) = \bar{x}$ for all $x \in \text{Int } R_+^n$ with $x \leq_K \bar{x}$, and $\omega(x) \subset [\bar{x}, \hat{x}]_K$ for all $x \in \text{Int } R_+^n$.

Proof. First, we consider the point $(0, w^0)$. In section 2, we saw that $\frac{\partial T_i}{\partial x_j}|_{(0, w^0)} = 0$ for any $i \in \bar{P}$ and $j \neq i$. Then there exists a neighborhood U of $(0, w^0)$ such that $\frac{\partial T_i}{\partial x_i}|_x > 1 + n\delta$, $-\delta < \frac{\partial T_i}{\partial x_j}|_x < \delta$ for all $x \in U$, $i \in \bar{P}$ and $j \neq i$, where δ is a sufficiently small positive number. Consider $(tz, w^0) \in U$ for $0 \leq t \leq 1$, where $z = \epsilon(1, 1, \dots, 1) \in \text{Int } R_+^{\#\bar{P}}$. For any $i \in \bar{P}$,

$$\begin{aligned} T_i((z, w^0)) &= T_i((z, w^0)) - T_i((0, w^0)) \\ &= \int_0^1 \text{grad} T_i(tz, w^0) dt \bullet (z, 0) \\ &> (1 + n\delta)\epsilon - n\delta\epsilon \geq \epsilon. \end{aligned}$$

On the other hand, $T((z, w^0)) \geq_K T((0, w^0)) = (0, w^0)$. Therefore, $T((z, w^0)) \geq_K (z, w^0)$. Hence, there is a fixed point $\bar{x} \geq_K (z, w^0)$ such that $\omega((z, w^0)) = \bar{x}$. The conclusion holds for all sufficiently small ϵ . The monotonicity of T implies that for

all $x \in \text{Int } R_+^n$ we have $(0, w^0) \leq_K x \leq_K \bar{x}$ and $\omega(x) = \bar{x}$. Similarly, there is a fixed point $\hat{x} \leq_K (u^0, 0)$ which attracts all $x \in \text{Int } R_+^n$ with $\hat{x} \leq_K x \leq_K (u^0, 0)$. Since $(0, w^0) \leq_K (u^0, 0)$, it is easy to see that $\bar{x} \leq_K \hat{x}$, and $\bar{x} \geq_K (z, w^0)$, $\hat{x} \leq_K (u^0, v)$, where $(z, w^0), (u^0, v) \in \text{Int } R_+^n$ and $\|z\|, \|v\|$ are sufficiently small. Hence $\bar{x}, \hat{x} \in \text{Int } R_+^n$. Now suppose that the additional assumptions hold. For all $x \in \text{Int } R_+^n$ with $x \geq_K \hat{x}$, we can choose $(u, 0) \in \text{Int } H_L^+$ with $(u^0, 0) \leq_K (u, 0)$ such that $x \leq_K (u, 0)$. Then $\hat{x} \leq_K \omega(x) \leq_K (u^0, 0) = \omega((u, 0))$. Because $\frac{\partial T_i}{\partial x_i}|_{(u^0, 0)} > 1$ for $i \in \bar{L}$, there is $(\bar{u}, 0)$ with $(\bar{u}, 0)_m - (u^0, 0)_m > 0$ for $m \in I$, $(\bar{u}, 0)_j - (u^0, 0)_j < 0$ for $j \in L \setminus I$ such that $\frac{\partial T_i}{\partial x_i}|_{(\bar{u}, 0)} > 1$ for $i \in \bar{L}$. By the monotonicity, $\frac{\partial T_i}{\partial x_i}|_{(u, 0)} \geq \frac{\partial T_i}{\partial x_i}|_{(\bar{u}, 0)} > 1$ for any $(u, 0) \in [\hat{x}, (\bar{u}, 0)]_K$. This implies that $\omega(x) \subset \text{Int } R_+^n$ for $x \in \text{Int } R_+^n$ with $x \geq_K \hat{x}$. Hence, $\omega(x) = \hat{x}$. Similarly, $\omega(x) = \bar{x}$ for $x \in \text{Int } R_+^n$ with $x \leq_K \bar{x}$. For any $x \in \text{Int } R_+^n$, there are $x^1, x^2 \in \text{Int } R_+^n$ satisfying $x^1 \leq_K \bar{x}$, $x^2 \geq_K \hat{x}$ such that $x^1 \leq_K x \leq_K x^2$. Hence $\omega(x) \subset [\bar{x}, \hat{x}]_K$. The proof is complete. \square

By this theorem and the conclusions in section 4, we can obtain the theorem for coexistence in a positive fixed point.

Theorem 6.2. *Suppose that T satisfies (AT), (BT) and (CT). Then there exists a unique fixed point $p \in \text{Int } R_+^n$ that is globally asymptotically stable with respect to $\text{Int } R_+^n$ if and only if the following conditions hold.*

- (1) *There exists $L \subset N$ with $I \subset L$ such that there exists a fixed point $(u^0, 0) \in \text{Int } H_L^+$ with $\frac{\partial T_i}{\partial x_i}|_{(u^0, 0)} > 1$ for any $i \in \bar{L}$.*
- (2) *There exists $P \subset N$ with $J \subset P$ such that there exists a fixed point $(0, w^0) \in \text{Int } H_P^+$ with $\frac{\partial T_j}{\partial x_j}|_{(0, w^0)} > 1$ for any $j \in \bar{P}$.*

Proof. Sufficiency. Suppose that the conditions (1) and (2) hold. By Lemma 4.4, $(u^0, 0)$ and $(0, w^0)$ attract $\text{Int } H_L^+$ and $\text{Int } H_P^+$ respectively. Thus Theorem 6.1 implies that there exists a fixed point $p \in \text{Int } R_+^n$. By the hypothesis (BT), p is locally asymptotically stable. By Lemma 4.4, p is globally asymptotically stable with respect to $\text{Int } R_+^n$.

Necessity. Assume that $p \in \text{Int } R_+^n$ is globally asymptotically stable with respect to $\text{Int } R_+^n$. By Theorem 3.1, there is a fixed point $a \in \text{Int } H_J^+$ with $a \leq_K p$ such that it attracts $\text{Int } H_J^+$. By Lemma 4.2 and the stability of p , $i(T, [a, p]_K, p) = 1$ and $i(T, [a, p]_K, q) = 0$ for any other fixed point $q \in [a, p]_K$. In particular, $i(T, [a, p]_K, a) = 0$. Lemma 4.2 implies that there is an index $i \in I$ such that $\frac{\partial T_i}{\partial x_i}|_a > 1$. If $\frac{\partial T_m}{\partial x_m}|_a > 1$ for any $m \in I$, then we choose $(0, w^0) = a \in \text{Int } H_J^+$ satisfying (2). Otherwise, applying Lemma 4.3 to $a, L = J$ and $l = i$, we get a fixed point $a^1 \in \text{Int } H_{L_1}^+$ with $L_1 = J \cup \{i\}$ such that it attracts $\text{Int } H_{L_1}^+$. Then $i(T, [a, p]_K, a^1) = 0$, and there exists $j \in I$ such that $\frac{\partial T_j}{\partial x_j}|_{a^1} > 1$. If $\frac{\partial T_r}{\partial x_r}|_{a^1} > 1$ for any $r \in I \setminus \{i\}$, then $(0, w^0) = a^1$ satisfies (2). Otherwise, we continue to apply Lemma 4.3 to a^1 . Obviously, after at most k steps, we obtain $P \subset N$ with $J \subset P$ such that there exists a fixed point $(0, w^0) \in \text{Int } H_P^+$ with $\frac{\partial T_j}{\partial x_j}|_{(0, w^0)} > 1$ for any $j \in \bar{P}$.

Similarly, there exists $L \subset N$ with $I \subset L$ such that there exists a fixed point $(u^0, 0) \in \text{Int } H_L^+$ satisfying $\frac{\partial T_i}{\partial x_i}|_{(u^0, 0)} > 1$ for any $i \in \bar{L}$. The proof is complete. \square

In nature it is often observed that the struggle for existence between two species competing for the same limited food supply and living space nearly always ends in the complete extinction of one of the species. This phenomenon is known as the

“principle of competitive exclusion”. Mathematically speaking, the species in the subcommunity L will ultimately become *extinct* if $(T^n(x))_i \rightarrow 0$ as $n \rightarrow +\infty$ for any $i \in L$.

Theorem 6.3. *Suppose that T satisfies (AT), (BT) and (CT). Then the subcommunity L will ultimately become extinct if and only if there is a boundary fixed point $p \in \text{Int } H_{\bar{L}}^+$ with $\bar{L} \neq N$ such that $\frac{\partial T_i}{\partial x_i}|_p < 1$ for any $i \in L$.*

Proof. The sufficiency follows immediately from Lemma 4.4. Suppose that the subcommunity L will ultimately become extinct. Then there cannot exist a positive fixed point. However, Theorem 4.1 tells us that there exists a fixed point p attracting all positive initial points. Therefore, by the definition of extinction, p lies in the boundary of R_+^n , that is, $p \in \text{Int } H_{\bar{L}}^+$ for some $L \subset N$. By (BT) and (CT), $\frac{\partial T_i}{\partial x_i}|_p < 1$ for any $i \in L$. The necessity follows. \square

Before ending this section, we note that Smith [7] studied permanence for autonomous ordinary differential equations (1.1) and obtained a permanent result in the case $L = I$ and $P = J$. Compared to his, the systems we have considered are more general and the conditions presented are much weaker.

7. APPLICATIONS

Consider the system of ordinary differential equations (1.1) in which x_i represents the population density of the i -th species and $f_j(x, t)$ represents the per capita growth rate of the j -th species. Assume that $f_j(x, t)$ is continuously differentiable in $R_+^n \times R$ for every $j \in N$.

The system (1.1) is called *cooperative* if, for every $(x, t) \in R_+^n \times R$,

$$(7.1) \quad \frac{\partial f_i}{\partial x_j}(x, t) \geq 0, \quad i \neq j,$$

and we say (1.1) is a system with *limited competition* if for every $(x, t) \in R_+^n \times R$,

$$(7.2) \quad \begin{cases} \frac{\partial f_i}{\partial x_j}(x, t) \geq 0, & \text{for } i, j \in I \text{ or } i, j \in J, \text{ and } i \neq j, \\ \frac{\partial f_i}{\partial x_j}(x, t) \leq 0, & \text{for } i \in I \text{ and } j \in J \text{ or } i \in J \text{ and } j \in I, \end{cases}$$

where $N = I \cup J$ and $I = \{1, 2, \dots, k\}$, $J = \{k+1, \dots, n\}$.

Now, we can conclude the following theorem.

Theorem 7.1. *Let $f(x, t) = (f_1(x, t), \dots, f_n(x, t))$ be continuously differentiable in $R_+^n \times R$, and 2π -periodic in t for fixed x . Assume that (7.1) or (7.2) holds and that every solution with $x(t_0) \geq 0$ of (1.1) can be continued to $[t_0, +\infty)$ and is bounded in R_+^n . If every 2π -periodic solution of (1.1) is hyperbolic (that is, its Floquet multipliers do not lie on the unit circle in the complex plane) and is asymptotically stable with respect to the face it belongs to, then (1.1) has a 2π -periodic solution $\varphi(t)$ which attracts all solutions whose initial points are in $\{x \geq 0 : x_i > 0 \text{ for } i \in N \text{ with } \varphi_i(0) > 0\}$. Furthermore, for a cooperative system (1.1), there exists a positive 2π -periodic solution that is globally asymptotically stable in $\text{Int } R_+^n$ if and only if there exists a 2π -periodic solution $\varphi(t) \in \text{Int } H_{\bar{L}}^+$ with $L \subset N$ such that all those Floquet multipliers corresponding to indices $j \in \bar{L}$ are larger than 1. Suppose that the system (1.1) has limited competition. Then (1.1) has a globally asymptotically stable positive 2π -periodic solution in $\text{Int } R_+^n$ if and only if*

(1) there is $L \subset N$ with $I \subset L$ such that (1.1) has a 2π -periodic solution in $\text{Int } H_L^+$ and all those Floquet multipliers corresponding to indices $j \in \overline{L}$ are larger than 1; and

(2) there is $P \subset N$ with $J \subset P$ such that (1.1) has a 2π -periodic solution in $\text{Int } H_P^+$ and all those Floquet multipliers corresponding to indices $i \in \overline{P}$ are larger than 1.

Corollary 7.2. *Let f be independent of t . Assume that all solutions of (1.1) are bounded in R_+^n . If every equilibrium is hyperbolic and asymptotically stable with respect to the face it belongs to, then there exists an equilibrium p which attracts all points in $\{x \geq 0 : x_i > 0 \text{ if } p_i > 0\}$. Furthermore, if (1.1) is an autonomous cooperative system, then it has a positive equilibrium p such that it is globally asymptotically stable if and only if there exists a boundary equilibrium $q \in \text{Int } H_L^+$ such that $f_i(q) > 0$ for $i \in \overline{L}$. If (1.1) is a system with limited competition, then it has a positive equilibrium that is globally asymptotically stable in $\text{Int } R_+^n$ if and only if*

(1) *there exists an equilibrium $p \in \text{Int } H_L^+$, where $L \subset N$ with $I \subset L$, such that $f_i(p) > 0$ for $i \in \overline{L}$; and*

(2) *there exists an equilibrium $q \in \text{Int } H_P^+$, where $P \subset N$ with $J \subset P$, such that $f_i(q) > 0$ for $i \in \overline{P}$.*

We remark that Corollary 7.2 covers a number of results in [7, 15, 16, 17, 25, 26], but the conditions are a little weaker and the result is more general.

The asymptotic behavior of solutions of the following classical Lotka-Volterra competition model with diffusion has been studied extensively:

$$(7.3) \quad \begin{cases} \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1(r_1 - a_{11}u_1 - a_{12}u_2) & \text{in } \Omega_T, \\ \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + u_2(r_2 - a_{21}u_1 - a_{22}u_2) & \text{in } \Omega_T, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\Omega_T, \\ u_1(x, 0) = u_{10}(x), u_2(x, 0) = u_{20}(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain of R^N with $N \geq 1$, $\partial\Omega$ and $\overline{\Omega}$ are the boundary and the closure of Ω , respectively, $\Omega_T = \Omega \times [0, T)$ and $\partial\Omega_T = \partial\Omega \times [0, T)$ for some $T \in (0, \infty]$, ν is the outward unit normal vector on $\partial\Omega$, r_i and a_{ij} ($i, j = 1, 2$) are all positive constants. The initial values u_{10} and u_{20} are non-negative continuous functions that are not identically zero.

The best results for (7.3) are summarized as follows (see [19], [20], [21]):

(I) when $r_1/r_2 > \max(a_{11}/a_{21}, a_{12}/a_{22})$, every solution of (7.3) $(u_1, u_2) \rightarrow (r_1/a_{11}, 0)$ uniformly as $t \rightarrow \infty$;

(II) when $r_1/r_2 < \min(a_{11}/a_{21}, a_{12}/a_{22})$, $(u_1, u_2) \rightarrow (0, r_2/a_{22})$ uniformly as $t \rightarrow \infty$;

(III) when $a_{11}/a_{21} > r_1/r_2 > a_{12}/a_{22}$, $(u_1, u_2) \rightarrow u^*$ uniformly as $t \rightarrow \infty$, where u^* is the solution of the equations $a_{11}u_1 + a_{12}u_2 = r_1$, $a_{21}u_1 + a_{22}u_2 = r_2$.

If the number of species is greater than 2, the asymptotic behavior for such systems is not clear so far. Applying our results above, we can give a perfect generalization for the results on two species stated above.

Now we consider

$$(7.4) \quad \begin{cases} \frac{\partial u}{\partial t} = \text{diag}(d_1, \dots, d_n) \Delta u + \text{diag}(u)(r + Mu) & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_T, \\ u(x, 0) = u^0(x) & \text{in } \Omega. \end{cases}$$

We assume that (7.4) satisfies the **Standard Assumptions**:

Ω is a bounded smooth domain of R^N with $N \geq 1$, $\partial\Omega$ and $\overline{\Omega}$ are the boundary and the closure of Ω , respectively, $\Omega_T = \Omega \times [0, T)$ and $\partial\Omega_T = \partial\Omega \times [0, T)$ for some $T \in (0, \infty]$, ν is the outward unit normal vector on $\partial\Omega$, $d_i > 0$ for $1 \leq i \leq n$, $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$, every component of the initial value u^0 is a non-negative continuous function, $r = (r_1, \dots, r_n)$ with every $r_i \neq 0$, and the constant matrix M has the form (1.2).

The asymptotic behavior of solutions for (7.4) is closely related to that of solutions of the Lotka-Volterra system of ordinary differential equations

$$(7.5) \quad \frac{du}{dt} = \text{diag}(u)(r + Mu).$$

For example, in any case of (I), (II) and (III), the equilibria of (7.5), which is a two-dimensional system, are hyperbolic and locally asymptotically stable relative to the faces they belong to. Now we can generalize it to the following.

Corollary 7.3. *Assume that (7.4) satisfies the Standard Assumptions. If every equilibrium of (7.5) is hyperbolic and asymptotically stable relative to the face it belongs to, then there is a constant steady state u^* which attracts every solution of (7.4) with initial value $u^0(x)$ satisfying that there exists $x \in \Omega$ such that $u_i^0(x) > 0$ for each i with $u_i^* > 0$.*

Moreover, there exists a positive steady state u^ which attracts all initial values $u^0(x)$ with the property that for each i , there exists $x \in \Omega$ such that $u_i^0(x) > 0$ if and only if the conditions (1) and (2) in Corollary 7.2 hold, where $f(u) = r + Mu$.*

Proof. Since we consider the asymptotic behavior of the system (7.4), T is surely ∞ . Theorem 7.3.1 in Smith [27] guarantees the existence, uniqueness and smoothness of the solution of (7.4). We also know if $u(x, t)$ is a solution of (7.4), then $u(x, t) \in R_+^n$ for any $(x, t) \in \overline{\Omega} \times [0, T)$.

By the result of Conway [28], if $u(x, t)$ is the solution of (7.4) and $u_i^0(x) > 0$ for some $x \in \Omega$, then $u_i(x, t) > 0$ for $(x, t) \in \overline{\Omega} \times (0, T)$. Therefore, without loss of generality, we may assume that either $u_i^0(x) \equiv 0$ on $\overline{\Omega}$ or $u_i^0(x) > 0$ on $\overline{\Omega}$ for each i , and furthermore, in order to prove this corollary, we can also assume that $u_i^0(x) > 0$ on $\overline{\Omega}$ for all i . Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be defined by

$$\begin{aligned} a_i &= \min_{x \in \overline{\Omega}} u_i^0(x) \text{ and } b_i = \max_{x \in \overline{\Omega}} u_i^0(x) \text{ for } i \in I, \\ a_j &= \max_{x \in \overline{\Omega}} u_j^0(x) \text{ and } b_j = \min_{x \in \overline{\Omega}} u_j^0(x) \text{ for } j \in J. \end{aligned}$$

Then $a, b \in \text{Int}R_+^n$, $a \leq_K u^0(x) \leq_K b$. By the maximum principle, we can prove that

$$\varphi_t(a) \leq_K u(x, t) \leq_K \varphi_t(b) \quad \text{for all } t > 0,$$

where $\varphi_t(u)$ denotes the solution flow of (7.5), which is obviously the solution of (7.4) with initial value u . The detailed proof can be seen in Theorem 7.3.4 of [27],

where the order is generated by the cone R_+^n . The proof is quite the same if the cone R_+^n is replaced by K . By assumption, there exists a constant steady state u^* such that $\lim_{t \rightarrow \infty} \varphi_t(a) = \lim_{t \rightarrow \infty} \varphi_t(b) = u^*$. This implies that $\lim_{t \rightarrow \infty} u(x, t) = u^*$. Thus the conclusions of Corollary 7.3 hold. \square

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